THE GENERALIZED SEGAL-BARGMANN TRANSFORM AND SPECIAL FUNCTIONS

MARK DAVIDSON AND GESTUR ÓLAFSSON

Abstract. Analysis of function spaces and special functions are closely related to the representation theory of Lie groups. We explain here the connection between the Laguerre functions, the Laguerre polynomials, and the Meixner-Pollacyck polynomials on the one side, and highest weight representations of Hermitian Lie groups on the other side. The representation theory is used to derive differential equations and recursion relations satisfied by those special functions.

Introduction

The broad subject of special functions is closely related to the representation theory of Lie groups and arises very naturally in analysis, number theory, combinatorics, and mathematical physics. The classical texts [51, 52] by Vilenkin and Vilenkin and Klimyk well document this interplay. The interested reader is also referred to [9] by Dieudonné and the recent text [1] by Andrews, Askey, and Roy. It is not our aim to add to this discussion in a general way, but mainly to concentrate on our specific work [5, 6, 7, 8] on the Meixner-Pollacyck polynomials, Laguerre functions, and Laguerre polynomials defined on symmetric cones. It is in the interplay among the various function spaces that model unitary highest weight representation for a Hermitian group that new results involving difference equations, differential equations, recursion formulas, and generating functions arise.

The classical Laguerre polynomials $L_n^\alpha(x)$ can be defined in several different ways. One of the oldest definitions is in terms of the generating function

$$(1-w)^{-\alpha-1}\exp\left(\frac{xw}{w-1}\right)=\sum_{n=0}^{\infty}L_n^\alpha(x)w^n\,,\quad |w|<1,\ -1<\alpha\,. \tag{0.1}$$

Another way is to use the Rodriguez type formula

$$L_n^\alpha(x)=\frac{e^x x^{-\alpha}}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha})\,. \tag{0.2}$$

Key words and phrases. Laguerre functions and polynomials, Meixner-Pollacyck polynomials, holomorphic discrete series, highest weight representations, bounded symmetric domains, real bounded symmetric domains, orthogonal polynomials, Laplace transform.

Research by G. Ólafsson supported by NSF grant DMS 0070607 and DMS 0139783.

Finally, they are special cases of hypergeometric functions:

$$L_n^\alpha(x) = \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} \frac{(-x)^k}{k!(n-k)!} \tag{0.3}$$

$$= \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} {}_1F_1(-n, \alpha+1; x)\,. \tag{0.4}$$

The polynomials

$$\sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}}\, L_n^\alpha(x), \quad n \in \mathbb{N},$$

form an orthonormal basis for the L^2-Hilbert space $L^2(\mathbb{R}^+, e^{-x}x^\alpha dx)$, or otherwise stated:

Theorem 0.1. *The functions*

$$\ell_n^\alpha(x) = L_n^\alpha(2x)e^{-x} \tag{0.5}$$

form an orthonormal basis for $L^2(\mathbb{R}^+, d\mu_\alpha)$*, where* $d\mu_\alpha(x) = x^\alpha\, dx$*.*

As was noted in [7], Theorem 3.1, a simple calculation shows that the functions

$$\{\ell_n^\alpha,\ n \in \mathbb{N}\}$$

have Laplace transform

$$\int_0^\infty e^{-xz}\ell_n^\alpha(x)\, d\mu_\alpha(x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} \left(\frac{z-1}{z+1}\right)^n (z+1)^{-(\alpha+1)}$$

which forms a basis for the space of SO(2)-finite vectors in a holomorphic discrete series of a conjugate version of $\mathrm{SL}(2,\mathbb{R})$. These representation theoretic relationships are the basic starting point in [8] and will be described in the following sections. In short let $\Omega \subset \mathbb{R}^d$ be a symmetric cone and let $T(\Omega) = \Omega + i\mathbb{R}^d$ be the corresponding tube domain. Let G^c be a connected semisimple Lie group, locally isomorphic to the connected component containing the identity of the group of holomorphic automorphisms of $T(\Omega)$. We use a natural orthogonal set of K-finite vectors in the Hilbert space, $\mathcal{H}_\nu(T(\Omega))$, of holomorphic functions corresponding to a highest weight representation of G^c and the generalized Segal-Bargmann transform, $\mathcal{L}_\nu$, defined in [39] to define an orthogonal set $\{\ell_m^\nu(x)\}_m$ on the space $L^2(T(\Omega), d\mu_\nu)$ for an appropriate measure $d\mu_\nu(x)$. A generalization of (0.3) was used in [15], p. 343, to define Laguerre polynomials, L_m^ν, for arbitrary symmetric cones and the following relationship was obtained:

$$\ell_m^\nu(x) = e^{-\operatorname{Tr}(x)} L_m^\nu(2x)\,. \tag{0.6}$$

Applying the same ideas to the bounded realization of $T(\Omega)$ gives a natural generalization of the Meixner-Pollacyck polynomials, [8], Section 5. Most of the main results in [6, 7, 8] concern the differential and difference relations among these special functions. Several authors have used the relation between highest weight modules and the Meixner-Pollacyck polynomials/Laguerre polynomials. In particular, we would like to mention [17, 18, 25, 28, 29, 33, 55, 56, 57] as related references, but as far as the authors know, the first time the Laguerre polynomials show up as coefficient functions

of a representation of a Lie group was in [51], p. 430–434, where Vilenkin relates the Laguerre polynomials to coefficient functions of representations of the group

$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & c & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c, d \in \mathbb{R}, c \neq 0 \right\} .$$

This overview article is based on the presentation of the second author at the conference *Representations of Lie Groups, harmonic analysis on homogeneous spaces and quantization*, at the Lorentz Center, University of Leiden, December 9 – 13, 2002. The second author would like to thank the organizers G. van Dijk and V.F. Molchanov, as well as the staff at the Lorentz Center for for the invitation and hospitality during his stay in Leiden.

1. Tube type domains

In this section we introduce the necessary notation and facts related to Hermitian Lie groups and bounded symmetric domains $\mathcal{D}$, containing the origin $0 \in \mathbb{C}^d$. Even if much of what we say is valid for all bounded symmetric domains in $\mathbb{C}^d$ we will assume from the beginning, as our applications will be restricted to that case, that $\mathcal{D}$ is isomorphic to a tube type domain $T(\Omega)$. There exists a connected semisimple Lie group that acts transitively on $\mathcal{D}$. For simplicity we will assume that G is simple. The general situation can be reduced to this case by considering direct products. Let $K = \{g \in G \mid g \cdot 0 = 0\}$. Then K is a maximal compact subgroup of G and $\mathcal{D} \simeq G/K$. Furthermore there exists an involution $\theta : G \to G$, a Cartan involution, such that

$$K = G^\theta = \{g \in G \mid \theta(g) = g\} .$$

In particular $\mathcal{D}$ is a *Riemannian symmetric space*. In this case $\mathbb{R}^d$ has the structure of an Euclidean Jordan algebra with identity element e, [15], and $\Omega = \{x^2 \mid x \in \mathbb{R}^d\}_0$ where the subscript $_0$ stands for the connected components containing e. The isomorphism $\mathcal{D} \simeq T(\Omega)$ is then given by the Cayley transform

$$\mathbf{c}(z) = (e+z)(e-z)^{-1} = \frac{e+z}{e-z}.$$

Let $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$ denote the complexification of $\mathfrak{g}$ and denote by $G_\mathbb{C}$ a simply connected and connected complex Lie group with Lie algebra $\mathfrak{g}_\mathbb{C}$. Then $g \cdot z$ is well defined for $g \in G_\mathbb{C}$ and almost all $z \in \mathbb{C}^d$ and $(g_1 g_2) \cdot z = g_1 \cdot (g_2 \cdot z)$ whenever defined. In particular there exists an element $c \in G_\mathbb{C}$ such that $c \cdot z = \mathbf{c}(z)$ for all $z \in \mathcal{D}$. Assuming that $G \subset G_\mathbb{C}$ we set $G^c = cGc^{-1}$. Then G^c acts transitively on $T(\Omega)$ and the stabilizer of $e \in \Omega$ is given by $K^c = cKc^{-1}$. Finally, there exists a conjugation $\sigma : \mathbb{C}^d \to \mathbb{C}^d$ which lifts to an involution $\tau : G_\mathbb{C} \to G_\mathbb{C}$ such that, with $H = G_0^\tau$, we have

$$\mathcal{D}_\mathbb{R} := \{z \in \mathcal{D} \mid \sigma(z) = z\}$$

is a totally real Riemannian subsymmetric space, isomorphic to $H/H \cap K$. We can choose σ so that the Cayley transform maps $\mathcal{D}_\mathbb{R}$ bijectively onto Ω. Notice that θ and τ commute and hence

$$\begin{aligned} \mathfrak{g} &= \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q} \\ &= \mathfrak{h}_k \oplus \mathfrak{h}_p \oplus \mathfrak{q}_k \oplus \mathfrak{q}_p \end{aligned}$$

where $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$, $\mathfrak{q} = \{X \in \mathfrak{g} \mid \tau(X) = -X\}$, and a subscript $_k$ (respectively $_p$) indicates intersection with $\mathfrak{k}$ (respectively $\mathfrak{p}$). Everything is now set up so that $\tau(g) = c^2(g)(c^{-1})^2$, $\mathfrak{g}^c = \mathrm{Ad}(c)\mathfrak{g}$, and the Lie algebra of G^c is given by

$$\mathfrak{g}^c = \mathfrak{h}_k \oplus i\mathfrak{h}_p \oplus i\mathfrak{q}_k \oplus \mathfrak{q}_p \,.$$

We note that $K^c := cKc^{-1} = H_{\mathbb{C}} \cap G^c$ and $H^c = cHc^{-1} = K_{\mathbb{C}} \cap G^c$.

The fact that $\mathcal{D}$ is an irreducible bounded symmetric domains implies that $\mathfrak{z}_{\mathfrak{k}}$, the center of $\mathfrak{k}$, is one dimensional. We can choose $Z_0 \in i\mathfrak{z}_{\mathfrak{k}}$ such that $\mathrm{ad}(Z_0) : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ has eigenvalues $0, 1$ and -1. The 0-eigenspace is exactly $\mathfrak{k}_{\mathbb{C}}$. Define $\mathfrak{p}^+$ to be the 1-eigenspace and $\mathfrak{p}^-$ to be the -1-eigenspace. Then $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are both abelian Lie algebras, normalized by $K_{\mathbb{C}}$, and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-$. Denote by $P^+ = \exp(\mathfrak{p}^+)$ the closed Lie subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{p}^+$ and similarly $P^- = \exp(\mathfrak{p}^-)$. Then $P^+ \times K_{\mathbb{C}} \times P^- \ni (p, k, q) \mapsto pkq \in G_{\mathbb{C}}$ is a diffeomorphism onto an open dense subset of $G_{\mathbb{C}}$. Furthermore $G \subset P^+K_{\mathbb{C}}P^-$. For $g \in P^+K_{\mathbb{C}}P^-$ we denote by $p(g) \in P^+$, $k_{\mathbb{C}}(g) \in K_{\mathbb{C}}$, and $q(g) \in P^-$ the inverse of the above diffeomorphism. Then the bounded realization of G/K inside $\mathfrak{p}^+$ is given by

$$G/K \ni gK \mapsto (\exp|_{\mathfrak{p}^+})^{-1}(p(g)) \in \mathfrak{p}^+ \,.$$

We will also need some basic information about roots. Let $\mathfrak{a}_k$ be a maximal abelian subspace of $\mathfrak{q}_k$. Then $\mathfrak{a}_k$ is in fact maximal abelian in $\mathfrak{q}$. Let Δ be the set of roots of $(\mathfrak{a}_k)_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ and let Δ_k be the set of roots of $(\mathfrak{a}_k)_{\mathbb{C}}$ in $\mathfrak{k}_{\mathbb{C}}$. Then

$$\Delta_k = \{\alpha \in \Delta \mid \alpha(Z_0) = 0\}$$

Set $\Delta_n = \{\alpha \in \Delta \mid \alpha(Z_0) \neq 0\}$. Then $\Delta_n = \Delta_n^+ \cup \Delta_n^-$, disjoint union, where

$$\Delta_n^{\pm} = \{\alpha \in \Delta \mid \alpha(Z_0) = \pm 1\} \,.$$

Recall that two roots $\alpha, \beta \in \Delta$ are called *strongly orthogonal* if $\alpha \pm \beta \notin \Delta$. In our situation there are only two root lengths. Choose a maximal set $\{\gamma_1, \ldots, \gamma_r\}$ of long strongly orthogonal roots in Δ_n^+. Notice that $r = \dim \mathfrak{a}_k$. The above set of roots can now be described as:

$$\Delta_n^{\pm} = \pm[\{\gamma_1, \ldots, \gamma_r\} \cup \{\frac{1}{2}(\gamma_i + \gamma_j) \mid i < j\}] \tag{1.1}$$

$$\Delta_k = \pm\{\frac{1}{2}(\gamma_i - \gamma_j) \mid i < j\}] \tag{1.2}$$

We choose the ordering such that $\gamma_1 > \ldots > \gamma_r > 0$. Then the positive set of roots are those with a '$+$' in the above equations. For $\alpha \in \Delta$ let

$$s_\alpha : \gamma \mapsto \gamma - \frac{2(\gamma, \alpha)}{(\alpha, \alpha)}\alpha$$

denote the corresponding Weyl group reflection. Here $(\cdot, \cdot)$ denotes a Weyl group invariant inner product on $i\mathfrak{a}_k^*$ such that $(\gamma_1, \gamma_1) = 1$. Notice that $s_{(\gamma_i - \gamma_j)/2}(\gamma_i) = \gamma_j$ and $s_{\gamma_i}((\gamma_i + \gamma_j)/2) = -(\gamma_i - \gamma_j)/2$. It follows that all the root spaces $(\mathfrak{g}_{\mathbb{C}})_{\gamma_j}$ have the same dimension, which is in fact one, and also the spaces $(\mathfrak{g}_{\mathbb{C}})_{(\gamma_i \pm \gamma_j)/2}$ have the same dimension, which we denote by a.

Let $H_j \in i\mathfrak{a}_k$ be such that $\gamma_i(H_j) = 2\delta_{ij}$. Then $i\mathfrak{a}_k = \oplus_{j=1}^r \mathbb{R}H_j$ and $Z = \frac{1}{2}(H_1 + \ldots + H_r)$. Let $\mathfrak{a} = \mathrm{Ad}(c)(\mathfrak{a}_k) = \mathrm{Ad}(c)^{-1}\mathfrak{a}_k$. Then $\mathfrak{a}$ is maximal abelian in $\mathfrak{h}_p$ and $\mathfrak{p}$.

The roots of $\mathfrak{a}$ in $\mathfrak{g}$ are given by $\Delta_{\mathfrak{a}} = \Delta \circ \mathrm{Ad}(c)$. We set:

$$\begin{array}{rcl} \xi_j &=& \mathrm{Ad}(c)^{-1} H_j \\ \xi &=& \xi_1 + \ldots + \xi_r = 2\mathrm{Ad}(c)^{-1} Z_0 \\ \beta_j &=& \gamma_j \circ \mathrm{Ad}(c)\,. \end{array}$$

Then $\mathfrak{a} = \bigoplus_{j=1}^r \mathbb{R}\xi_j$ and ξ is central in $\mathfrak{h}$. We use $\beta_1, \ldots, \beta_r$ to identify $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}^r$ by

$$(\lambda_1, \ldots, \lambda_r) \mapsto \lambda_1\beta_1 + \ldots + \lambda_r\beta_r\,.$$

We also embed $\mathbb{C}$ into $\mathfrak{a}_{\mathbb{C}}^*$ by $\lambda \mapsto \lambda(\beta_1 + \ldots + \beta_r)$.

If G is simple, then by the classification the triple (G, K, H) is locally isomorphic to one of the following:

G	K	H	$K \cap H$	r	a
$\mathrm{Sp}(n, \mathbb{R})$	$\mathrm{U}(n)$	$\mathrm{GL}(n, \mathbb{R})_+$	$\mathrm{O}(n)$	n	1
$\mathrm{SU}(n, n)$	$\mathrm{U}(n) \times \mathrm{U}(n)$	$\mathrm{GL}(n, \mathbb{C})_+$	$\mathrm{U}(n)$	n	2
$\mathrm{SO}^*(4n)$	$\mathrm{U}(2n)$	$\mathrm{SU}^*(2n)\mathbb{R}^+$	$\mathrm{Sp}(2n)$	$2n$	4
$\mathrm{SO}(2, k)$	$\mathrm{S}(\mathrm{O}(2) \times \mathrm{O}(k))$	$\mathrm{SO}(1, k-1)\mathbb{R}^+$	$\mathrm{SO}(k-1)$	$2n$	$k-2$
$\mathrm{E}_{7(-25)}$	$\mathrm{E}_6\mathbb{T}$	$\mathrm{E}_{6(-26)}\mathbb{R}^+$	F_4	3	8

Here the subscript $_+$ stands for the real and positive determinant.

Example 1.1 (Symmetric Matrices)**.** Let $\mathbb{R}^d = \mathrm{Sym}(n, \mathbb{R})$, $d = n(n+1)/2$, be the real space of $n \times n$ real symmetric matrices. The complexification of $\mathbb{R}^d$ is then the space of $n \times n$ complex symmetric matrices $\mathrm{Sym}(n, \mathbb{C})$. Obviously $\mathrm{Sym}(n, \mathbb{C})$ is a complex Jordan algebra, where the product is given by $X \cdot Y := (XY + YX)/2$. Furthermore,

$$\Omega = \{X \in \mathrm{Sym}(n, \mathbb{R}) \mid X > 0\}\,,$$

where $>$ stands for positive definite. The action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathrm{Sym}(n, \mathbb{R})$ is given by

$$g \cdot X = gXg^T\,.$$

The orbits are parametrized by the signature. In particular, the cone of positive definite matrices is homogeneous. The identity element is just the usual identity matrix $I_n \in \Omega$ and the stabilizer of I_n is $\mathrm{SO}(n)$.

Write an element $g \in \mathrm{Sp}(n, \mathbb{R})$ as

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\,.$$

Then the action of $\mathrm{Sp}(n, \mathbb{R})$ on $T(\Omega)$ is given by

$$g \cdot Z = (AZ + B)(CZ + D)^{-1}\,. \tag{1.3}$$

Notice that (1.3) in fact defines an almost everywhere defined action of $\mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}} = \mathrm{Sp}(n, \mathbb{C})$ on $\mathrm{Sym}(n, \mathbb{C})$. We have

$$\mathcal{D} = \{X \in \mathrm{Sym}(n, \mathbb{C}) \mid I_n - Z^*Z > 0\}$$

and the Cayley transform is given by

$$c(Z) = (I_n + Z)(I_n - Z)^{-1} = 2^{-2n} \begin{pmatrix} I_n & I_n \\ -I_n & I_n \end{pmatrix} \cdot Z\,.$$

Define $\sigma : \mathrm{Sym}(n, \mathbb{C}) \to \mathrm{Sym}(n, \mathbb{C})$ to be the usual complex conjugation. Then $\mathcal{D}_{\mathbb{R}} = \{Z \in \mathrm{Sym}(n, \mathbb{R}) \mid I_n - Z^2 > 0\}$ and a simple calculation shows that $\mathbf{c}(X) \in \Omega$ for $X \in \mathcal{D}_{\mathbb{R}}$ as was to be expected.

The case $\mathbb{R}^d = \{Z \in \mathrm{M}(n, \mathbb{C}) \mid Z^* = Z\}$, $G = \mathrm{SU}(n, n)$ and $H = \mathrm{GL}(n, \mathbb{C})_+$ is treated is the same way. Notice that in this case the complexification of $\mathbb{R}^d$ is given by $\mathbb{R}^d_{\mathbb{C}} = \mathrm{M}(n, \mathbb{C})$ as every complex matrix can be written in an unique way as $Z = X+iY$ with $X = X^*$ and $Y + Y^*$. Simply set $X = 1/2(Z + Z^*)$ and $Y = 1/(2i)(Z - Z^*)$.

We refer to [15, 32, 50] for further information on the structure of bounded symmetric domains and the relation between bounded symmetric domains and Jordan algebras. For the more group theoretical description we refer to [21, 30, 31]. The connection between Hermitian symmetric spaces and causal symmetric spaces G/H is described in [23].

2. Highest weight representations

The unitary highest weight representations of G are well understood unitary representations that can be realized in a Hilbert space of holomorphic functions on $\mathcal{D}$. The idea described in this section is to use the *Restriction principle*, introduced in [39] (see also [36, 41]), to construct the Berezin transform and generalized Segal-Bargmann transform and to transfer information known from the highest weight representations to extract information about $L^2(\mathcal{D}_{\mathbb{R}})$ and $L^2(\Omega, d\mu_\nu)$, where μ_ν is a measure on Ω to be introduced in a moment. Thus: *We want to use the highest weight representations to do harmonic analysis on bounded real domains and symmetric cones.* It should be noted that almost all Riemannian symmetric spaces can be realized as a real form of a Hermitian symmetric spaces. We will in this section give a brief introduction to the theory of highest weight representations. We refer to [13, 14, 20, 22, 24, 34, 40, 46, 45, 53] for more information.

Let $p = 2d/r$, and let $J(g, z)$ be the complex Jacobian determinant of the action of G on $\mathcal{D}$. If we are discussing the tube type realization, then we will use the same notation for the complex Jacobian determinant of the action of G^c on $T(\Omega)$. The map $(g, z) \mapsto J(g, z)$ can be expressed in terms of the $K_{\mathbb{C}}$-projection $(g, z) \mapsto k_{\mathbb{C}}(g \exp(z)) \in K_{\mathbb{C}}$ and

$$\rho_n = \frac{1}{2} \sum_{\alpha \in \Delta_n} \dim((\mathfrak{g}_{\mathbb{C}})_\alpha)\alpha = \frac{1}{2}\left(1 + \frac{a(r-1)}{2}\right)(\gamma_1 + \ldots + \gamma_r)$$

as

$$J(g, z) = \det \mathrm{Ad}(k_C(g \exp z))|_{\mathfrak{p}^+} = k_C(g \exp z)^{2\rho_n}\,.$$

Let Δ be the determinant function on the Jordan algebra, $\mathbb{R}^d$, and Tr the trace functional. In the case $\mathbb{R}^d = \mathrm{Sym}(n, \mathbb{R})$ this is just the usual determinant function and trace. We denote by Δ_j the principal minors. Recall that $\Delta_r = \Delta$. For $(\alpha_1, \ldots, \alpha_r) \in \mathbb{C}^r$ let

$$\Delta_\alpha(w) := \Delta_1(w)^{\alpha_1 - \alpha_2} \Delta_2(w)^{\alpha_2 - \alpha_3} \ldots \Delta_r(w)^{\alpha_r} \tag{2.1}$$

and

$$\psi_\alpha(x) = \int_{K \cap H} \Delta_\alpha(kx)\, dk\,. \tag{2.2}$$

Let $h(z,w) = \Delta(e - z\bar{w})$. Finally, we define the Gindikin-Koecher gamma function by

$$\Gamma_\Omega(\lambda) := \int_\Omega e^{-\operatorname{Tr}(x)} \Delta_\lambda(x) \Delta^{-d/r}\, dx, \quad \lambda \in \mathbb{C}\,. \tag{2.3}$$

We have

$$\Gamma_\Omega(\lambda) = (2\pi)^{(d-r)/2} \prod_{j=1}^{r} \Gamma(\lambda_j - \frac{a}{2}(j-1))\,,$$

where Γ is the usual Γ-function. For $m \in \Lambda := \{(m_1, \ldots, m_r) \in \mathbb{N}_0^r \mid m_1 \geq m_2 \geq \ldots \geq m_r\}$ denote by τ_m the irreducible $K \cap H$-spherical representation with lowest weight $-m$. Denote by $P(\mathbb{C}^r)$ the space of polynomial functions on $\mathbb{C}^r$, Then τ_m can be realized (with multiplicity one) in a subspace $P_m(\mathbb{C}^r) \subset P(\mathbb{C}^r)$. Furthermore (see [47])

$$P(\mathbb{C}^r) = \bigoplus_{m \in \Lambda} P_m(\mathbb{C}^r)\,.$$

There is a canonical series of highest weight representations (or weighted Bargmann spaces in some cases)

$$(\pi_\nu, \mathbf{H}_\nu(\mathcal{D})) \quad \nu \in \left\{0, \frac{a}{2}, \ldots, \frac{a}{2}(r-1)\right\} \cup (\frac{a}{2}(r-1), \infty),$$

the so-called Berezin-Wallach set. To simplify the notation for the moment write $\mathbf{H}_\nu$ for $\mathbf{H}_\nu(\mathcal{D})$. The space $\mathbf{H}_\nu$ is a Hilbert space of holomorphic functions on $\mathcal{D}$. For $\nu > a(r-1)+1$ the norm is given by

$$\|F\|_\nu^2 = \alpha_\nu \int_{\mathcal{D}} |F(z)|^2 h(z,z)^{\nu - p}\, dz, \quad \alpha_\nu = \frac{1}{\pi^d} \frac{\Gamma_\Omega(\nu)}{\Gamma_\Omega(\nu - d/r)}\,.$$

In this case, if we assume that the center of G is finite, the representation $(\pi_\nu, \mathbf{H}_\nu)$ can be realized as a discrete summand in $L^2(G)$ ([20]) and $L^2(G/H)$ ([37, 38]). For $\nu < a(r-1)+1$ we use analytic continuation of this norm. The representation of G on $\mathbf{H}_\nu$ is given by

$$\pi_\nu(g)F(z) = J(g^{-1}, z)^{\frac{\nu}{p}} F(g^{-1}z)\,. \tag{2.4}$$

The facts that we will need are:

Theorem 2.1. *Let the notation be as above. Then the following holds:*

(1) $\mathbf{H}_\nu$ *is a reproducing kernel Hilbert space with reproducing kernel*

$$K_w(z) = K(z,w) = h(z,w)^{-\nu} = \Delta(e - z\bar{w})^{-\nu}.$$

(2) *If* $\nu > (r-1)\frac{a}{2}$ *then the space of polynomials* $P(\mathbb{C}^d)$ *is dense in* $\mathbf{H}_\nu$*. More specifically, the space of* K*-finite vectors* $(\mathbf{H}_\nu)_K$ *can be naturally identified with* $P(\mathbb{C}^d)$

(3) *All of the* K*-representations* $(\tau_m, P_m(\mathbb{C}^d))$ *are* $K \cap H$*-spherical and*

$$P_m(\mathbb{C}^d)^{K \cap H} = \mathbb{C}\psi_m.$$

(4) $\mathbf{H}_\nu^{K \cap H} \simeq \bigoplus_{m \in \Lambda} \mathbb{C}\psi_m$.

(5) *The norm of the function* ψ_m *is given by*

$$\|\psi_m\|^2 = d_m^{-1}\frac{\left(\frac{d}{r}\right)_m}{(\nu)_m}$$

where $d_m = \dim P_m(\mathbb{C}^d)$.

Notice that (1) means that the point evaluations are continuous linear functionals on $\mathbf{H}_\nu$ and

$$F(z) = (F|K_z) \quad \text{for all } F \in \mathbf{H}_\nu.$$

3. The Berezin transform and generalized Segal-Bargmann transform

In this section we recall the basic facts about the *Berezin transform* and *generalized Segal-Bargmann transform*. We refer to [2, 12, 10, 11, 19, 27, 35, 36, 39, 41, 48, 49] for further information. We define a map, *the restriction map*, (see [39]) $R_\nu : \mathbf{H}_\nu \to C^\infty(\mathcal{D}_\mathbb{R})$ by

$$R_\nu F(x) = h(x)^{\frac{\nu}{2}} F(x),$$

where $h(x) = h(x, x)$. Then R_ν is injective. Since the invariant measure $d\eta$ on $\mathcal{D}_\mathbb{R}$ is $d\eta(x) = h(x)^{-\frac{p}{2}}\, dx$ (and a few other things) we get

Lemma 3.1. *$R_\nu F \in L^2(\mathcal{D}_\mathbb{R}, d\eta)$ for all $F \in P(\mathbb{C}^d)$ if and only if $\nu > \frac{a}{2}(r-1)$.*

Thus the restriction map

$$R_\nu : (\mathbf{H}_\nu)_K = P(\mathbb{C}^d) \to L^2(\mathcal{D}_\mathbb{R}, d\eta)$$

is defined for all ν in the continuous part of the Berezin-Wallach set. We collect the important properties in the following theorem (see [8], section 3):

Theorem 3.2. *Assume that $\nu > \frac{a}{2}(r-1)$. Then the following hold:*

(1) *$R_\nu(P(\mathbb{C}^d))$ is dense in $L^2(\mathcal{D}_\mathbb{R}, d\eta)$.*
(2) *$R_\nu : \mathbf{H}_\nu \to L^2(\mathcal{D}_\mathbb{R}, d\eta)$ is closed.*
(3) *The Berezin transform $R_\nu R_\nu^*$ is given by*

$$R_\nu R_\nu^* f(y) = \int_{\mathcal{D}_\mathbb{R}} \frac{h(y)^{\frac{\nu}{2}} h(x)^{\frac{\nu}{2}}}{h(y,x)^\nu} f(x)\, d\eta(x) = D_\nu \star f(h) \quad (y = h \cdot 0),$$

where $D_\nu(h) = J(h, 0)^{\frac{\nu}{p}}$.

(4) *If $\nu > a(r-1)$ then $R_\nu R_\nu^* : L^2(\mathcal{D}_\mathbb{R}, d\eta) \to L^2(\mathcal{D}_\mathbb{R}, d\eta)$ is continuous with norm $||R_n R_n^*||_2 \le ||D_\nu||_{L^1} < \infty$.*
(5) *If $\nu > a(r-1)$ then $R_\nu R_\nu^* : L^\infty(\mathcal{D}_\mathbb{R}, d\eta) \to L^\infty(\mathcal{D}_\mathbb{R}, d\eta)$ is continuous with norm $||R_n R_n^*||_\infty \le ||D_\nu||_{L^1} < \infty$.*

We are thus able to apply $R_\nu R_\nu^*$ to bounded spherical functions! The proof of (3) is standard:

$$\begin{aligned} R_\nu^* f(z) &= (R_v^* f|K_z) \\ &= (f|R_v K_z) \\ &= \int f(x) h(x)^{\frac{\nu}{2}} h(y,x)^{-\nu}\, d\eta. \end{aligned}$$

This implies
$$R_\nu R_\nu^* f(y) = \int f(x) h(y)^{\frac{\nu}{2}} h(x)^{\frac{\nu}{2}} h(y,z)^{-\nu}\, d\eta.$$
In particular, this shows that the restriction principle defined in [39] results in the Berezin transform.

For $\nu > \frac{a}{2}(r-1)$, the restriction map $R_\nu : \mathbf{H}_\nu \to L^2(\mathcal{D}_\mathbb{R}, d\eta)$ is closed with dense image. We can therefore define a unitary isomorphism $U_\nu : L^2(\mathcal{D}_\mathbb{R}, d\eta) \to \mathbf{H}_\nu$ by polarization. Thus U_ν satisfies
$$R_\nu^* = U_\nu \sqrt{R_\nu R_\nu^*}.$$
The map U_ν is the generalized Segal-Bargmann transform. Let $c(\nu)$ denote the *Harish-Chandra* c-function, W the Weyl group corresponding to the root system $\Delta(\mathfrak{a}, \mathfrak{h})$, $w = \#W$, and $\mathcal{F}$ the Harish -Chandra/ Helgason spherical Fourier transform
$$\begin{aligned} \mathcal{F} : L^2(\mathcal{D}_\mathbb{R}, d\eta)^{H\cap K} &\to L^2(\mathfrak{a}, \frac{1}{\omega}\frac{d\lambda}{|c(\lambda)|^2})^W \\ F &\mapsto \int_{\mathcal{D}_\mathbb{R}} F(x)\varphi_\lambda(x) d\eta(x) \end{aligned}$$
where φ_λ is the spherical functions on $H/H\cap K \simeq \mathcal{D}_\mathbb{R}$ given by
$$\varphi_\lambda(x) = \psi_{i\lambda+\rho}((e+x)(e-x)^{-1}) = \psi_{i\lambda+\rho}(\mathbf{c}(x))\,.$$
Here ρ is half the sum of the positive roots for $H/(H\cap K)$. Then the above discussion results in a unitary isomorphism
$$\mathcal{F}\circ U_\nu^* : P(\mathbb{C}^r)^{K\cap H} = \mathbf{H}_\nu^{H\cap K} \to L^2(\mathfrak{a}, \frac{1}{w}\frac{d\lambda}{|c(\lambda)|^2})^W$$
and we get:

Theorem 3.3. *If $\nu > \frac{a}{2}(r-1)$ then the functions*
$$\{\mathcal{F}\circ U_\nu^*(\psi_m)\}_{\mathbf{m}\geq 0}$$
form an orthogonal basis for $L^2(\mathfrak{a}, \frac{1}{w}\frac{d\lambda}{|c(\lambda)|^2})^W$.

4. The Polynomials $p_{\nu,m}(\lambda)$

The next obvious task is to understand the functions $\mathcal{F}\circ U_\nu^*(\psi_\mathbf{m})$. For that we define the polynomials $p_{\nu,\mathbf{m}}(\lambda)$ by
$$p_{\nu,m}(\lambda) = \|\psi_\mathbf{m}\|^{-2}\psi_m(\partial_x)[\Delta(e-x^2)^{-\frac{\nu}{2}}\phi_\lambda(x)]_{x=0}, \tag{4.1}$$
We have:

Lemma 4.1 ([8], Lemma 4.1)**.**
$$\Delta(e-x^2)^{-\frac{\nu}{2}}\phi_\lambda(x) = \sum_m p_{\nu,m}(\lambda)\psi_m(x).$$

We define the constants c_ν and b_ν by
$$c_\nu = \|D_\nu\|_{L^1} \quad \text{and} \quad R_\nu R_\nu^*(\varphi_\lambda) = c_\nu^{-1} b_\nu(\lambda)\varphi_\lambda\,.$$
The numbers c_ν and $b_\nu(\lambda)$ have been evaluated by Zhang, [56].

Theorem 4.2 ([8], Proposition 4.3)**.** *Assume* $\nu > \frac{a}{2}(r-1)$. *Then*

$$\mathcal{F}(U_\nu^*\psi_{\mathbf{m}})(\lambda) = c_\nu^{-\frac{1}{2}}\sqrt{b_\nu(\lambda)}\|\psi_{\mathbf{m}}\|_\nu^2 p_{\nu,\mathbf{m}}(\lambda)\,.$$

The numbers c_ν and $b_\nu(\lambda)$ have been computed by Zhang.

5. Example: $G = SU(1,1)$

Let $\mathcal{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. Whenever defined we write for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C})$ and $z \in \mathcal{D}$:

$$g \cdot z = \frac{az+b}{cz+d}\,.$$

Then, restricted to $\mathrm{SU}(1,1)$, this defines a transitive action of $\mathrm{SU}(1,1)$ on $\mathcal{D}$. In this case we take the conjugation σ as the usual complex conjugation $z \mapsto \bar{z}$. Then $\mathcal{D}_\mathbb{R} = (-1,1)$ and $\psi_m(x) = x^m$. Notice that

$$H = \left\{ h_t = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

and $h_t(0) = \tanh(t)$. In particular $K \cap H = \{\mathrm{Id}\}$ is trivial. As $(1+x)(1-x)^{-1}$ is real and positive for all $x \in (-1,1)$ we can define

$$G_{\nu,\lambda}(x) = (1-x^2)^{-\frac{\nu}{2}}(\frac{1+x}{1-x})^{i\lambda}\,.$$

Then we can expand $G_{\nu,\lambda}$ as

$$G_{\nu,\lambda}(x) = \sum_{n=0}^{\infty} p_{n,\nu}(\lambda)x^n,$$

with $p_{n,\nu}(\lambda) = (\frac{\nu}{2} + i\lambda)_n\, {}_2F_1(\frac{\nu}{2} - i\lambda, -n, -\frac{\nu}{2} - i\lambda - n + 1, 1)$, the Meixner-Pollacyck polynomials.

The Hilbert space $\mathbf{H}_\nu$ is given as the space of holomorphic functions $F : \mathcal{D} \to \mathbb{C}$ such that

$$\|F\|^2 = \frac{1}{\pi}\frac{\Gamma(\nu)}{\Gamma(\nu-1)} \int |F(x+iy)|^2(1-(x^2+y^2))^{2\nu-2}dxdy$$

and, whenever defined for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have

$$\pi_\nu(g)F(z) = (-bz+a)^{-2\nu}f(g\cdot z)$$

The connection to the representation π_ν is as follows. We have $\xi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{h}$ and $Z_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{k}$. Thus

$$\begin{aligned} \pi_\nu(\xi)f(x) &= \nu x f(x) - (1-x^2)f'(x)\,, \\ \pi_\nu(\xi)z^m &= (\nu+m)z^{m+1} - mz^{m-1}\,, \\ \pi_\nu(\xi)G_{\nu,\lambda} &= -2i\lambda G_{\nu,\lambda}\,. \end{aligned}$$

The last item follows from the simple calculation

$$2i\lambda p_{\nu,n}(\lambda) = (n+1)p_{\nu,n+1}(\lambda) - (\nu+n-1)p_{\nu,n-1}(\lambda)\,. \tag{5.1}$$

Now using the following facts that

$$\begin{aligned} \pi_\nu(-Z_0)f(x) &= xf'(x)\,, \\ \pi_\nu(-Z_0)z^m &= mx^m\,, \\ xG'_{\nu,,\lambda}(x) &= \sum_{m=0}^{\infty} mp_{m,\nu}(x)x^m\,. \end{aligned}$$

we get

$$-(\nu+2n)p_{\nu,n}(\lambda) = (-\frac{\nu}{2}+i\lambda)p_{\nu,n}(\lambda+i) - (\frac{\nu}{2}+i\lambda)p_{\nu,n}(\lambda-i)\,. \tag{5.2}$$

6. Difference Relations

The general recurrence relation corresponding to (5.1) and difference relation generalizing (5.2) are as follows: Choose $Z_0 \in \mathfrak{z}\mathfrak{k}_{\mathbb{C}}$ as before, i.e., such that $\mathrm{ad}(Z_0)$ has eigenvalues $0, 1$, and -1. As before set $\xi = \mathrm{Ad}(c)(-2Z_0) = \mathrm{Ad}(c)^{-1}(2Z_0)$. Denote by e_k the standard basis vector of $\mathbb{R}^r$ with a 1 in each k-th position and 0's elsewhere. For n an r-tuple define

$$\binom{n}{n-e_k} := (n_k + \frac{a}{2}(r-k))\prod_{j\neq k}\frac{n_k - n_j + \frac{a}{2}(j-k-1)}{n_k - n_j + \frac{a}{2}(j-k)}$$

and

$$c_n(k) = \prod_{j\neq k}\frac{n_k - n_j - \frac{a}{2}(j-k-1)}{n_k - n_j - \frac{a}{2}(j-k)}\,.$$

Then

Theorem 6.1 ([8],Theorem 5.2)**.** *With the above notation we have:*

$$\pi_\nu(-\xi)\psi_m = \sum_{j=1}^{r}\binom{m}{m-e_j}\psi_{m-e_j} - \sum_{j=1}^{r}(\nu + m_j - \frac{a}{2}(j-1))c_m(j)\psi_{m+e_j}.$$

Let $q_{m,\nu}(z) = \Delta(z+e)^{-\nu}\psi_m((z-e)(z+e)^{-1})$. Since $\xi = -2\mathrm{Ad}(Z_0)$ and Z_0 is central in $\mathfrak{k}$ it follows, by some calculations, that:

$$\begin{aligned} \pi_\nu(\xi)q_{m,\nu} &= (r\nu + 2\|m|)q_{m,\nu} \\ \pi_\nu(-2Z_0)q_{m,\nu} &= \sum_{j=1}^{r}\binom{m}{m-e_j} q_{m-e_j,\nu} - \sum_{j=1}^{r}(\nu + n_j - \frac{a}{2}(j-1))c_m(j)q_{m+e_j,\nu}. \end{aligned}$$

Putting those pieces together we get:

Theorem 6.2 ([8], Theorem 5.6)**.** *We have the following difference relations amongst the $p_{\nu,m}(\lambda)$:*

$$2\sum_{j=1}^{r}(i\lambda_j+\rho_j)p_{\nu,m}(\lambda) = \sum_{j=1}^{r}\binom{m+e_j}{m}p_{\nu,m+e_j}(\lambda) - (\nu+m_j-1-\frac{a}{2}(j-1))c_{m-e_j}p_{\nu,m-e_j}(\lambda)\,.$$

Theorem 6.3 ([8], Theorem 6.1)**.** *We have*

$$-(r\nu + 2|m|)p_{\nu,m}(\lambda) \;=\; \sum_{j=1}^{r}\binom{i\lambda+\rho-\frac{\nu}{2}}{i\lambda+\rho-\nu 2 - e_j}p_{\nu,m}-$$

$$\sum_{j=1}^{r}(\frac{\nu}{2}+i\lambda_j+\rho_j-\frac{a}{2}(j-1))c_{i\lambda+\rho-\frac{\nu}{2}}(j)p_{\nu,m}(\lambda-ie_j).$$

7. The Unbounded Realization

In this section we discuss the unbounded realization $\mathcal{D}\simeq T(\Omega)=i\mathbb{R}^d+\Omega$. This can be used to study recurrence and differential equations for Laguerre polynomials and functions on Ω. We refer to [8, 15, 44] for further references.

From now on $\nu\in\mathbb{C}$ is identified with $\nu(\beta_1+\ldots+\beta_r)\in\mathfrak{a}^*$. Using the Cayley transform we get a space of holomorphic functions on $T(\Omega)$, $\mathbf{H}_\nu(T(\Omega))=\pi_\nu(c)\mathbf{H}_\nu$, and get an orthonormal basis

$$q_{m,\nu}(z)=\Delta(z+e)^{-\nu}\psi_m(\frac{z-e}{z+e})$$

for $\widetilde{\mathbf{H}}_\nu^{H\cap K}$. Notice that these functions correspond to the functions $z\to\frac{(z-i)^m}{(z+i)^{\nu+m}}$ on the upper half plane in the case $G=\mathrm{SL}(2,\mathbb{R})$. Let us describe this in more detail, as the standard normalization for the unbounded realization is usually different from the one in the bounded realization. For $\nu>1+a(r-1)$ let $\mathbf{H}_\nu(T(\Omega))$ be the space of holomorphic functions $F:T(\Omega)\to\mathbb{C}$ such that

$$\|F\|_\nu^2:=\beta_\nu\int_{T(\Omega)}|F(x+iy)|^2\Delta(y)^{\nu-2d/r}\,dxdy<\infty \tag{7.1}$$

where

$$\beta_\nu=\frac{2^{r\nu}}{(4\pi)^d\Gamma_\Omega(\nu-d/r)}\,. \tag{7.2}$$

Then $\mathbf{H}_\nu(T(\Omega))$ is a non-trivial Hilbert space. For $\nu\le 1+a(r-1)$ this space reduces to $\{0\}$ and analytic continuation is again used to define the norm [44]. If $\nu=2d/r$ this is the *Bergman space*.

Instead of using the H^c-invariant measure on the cone Ω we use a weighted measure $d\mu_\nu(x)=\Delta(x)^{\nu-d/r}dx$ corresponding to the measure $x^{\nu-1}dx$ on $\mathbb{R}^+$. Thus we get the weighted L^2-spaces

$$L^2_\nu(\Omega)=L^2(\Omega,\Delta(x)^{\nu-\frac{d}{r}}\,dx).$$

In this normalization the restriction map is simply $R_\nu F=F|_\Omega$ and the unitary part becomes the Laplace transform $\mathcal{L}_\nu$. We define $\mathcal{L}_\nu$ on the domain

$$\{f\in L^2_\nu(\Omega,d\mu_\nu)\mid\int_\Omega e^{-(\omega,x)}|f(x)|d\mu_\nu(x)<\infty\}$$

for all $\omega\in\Omega$ by

$$\mathcal{L}_\nu(f)(z):=\int_\Omega e^{-(z,x)}f(x)d\mu_\nu(x)\,.$$

Note that by the Cauchy-Schwarz inequality, the condition $f\in L^2_\nu(\Omega)$ implies, since $|e^{-(s+it,x)}|=e^{-(s,x)}$, that $\mathcal{L}_\nu f$ is a well-defined function on $T(\Omega)$. To simplify notation we sometimes write R for R_ν and $\mathcal{L}$ for $\mathcal{L}_\nu$. Denote by $\mathcal{L}^\Omega=R_\nu\circ\mathcal{L}_\nu$. Then $\mathcal{L}^\Omega$ is a self-adjoint positive operator $L^2(\Omega,d\mu_\nu)\to L^2(\Omega,d\mu_\nu)$.

Theorem 7.1. *Let the notation be as above. Assume that $\nu>1+a(r-1)$. Then the following hold:*

(1) *The space* $\mathbf{H}_\nu(T(\Omega))$ *is a reproducing kernel Hilbert space.*
(2) *The map*
$$\Psi_\nu := \frac{1}{\sqrt{\Gamma_\Omega(\nu)}}\,\pi_\nu(c)^{-1} : \mathbf{H}_\nu(T(\Omega)) \to \mathbf{H}_\nu(\mathcal{D})$$
is a unitary isomorphism.
(3) *The reproducing kernel of* $\mathbf{H}_\nu(T(\Omega))$ *is given by*
$$K_\nu(z,w) = \Gamma_\Omega(\nu)\Delta\,(z+\bar{w})^{-\nu}$$
(4) *If* $\nu > (r-1)\frac{a}{2}$ *then there exists a Hilbert space* $\mathbf{H}_\nu(T(\Omega))$ *of holomorphic functions on* $T(\Omega)$ *such that* $K_\nu(z,w)$ *defined in (2) is the reproducing kernel of that Hilbert space and the universal covering group of* G^c *acts unitarily and irreducibly on* $\mathbf{H}_\nu(T(\Omega))$*.*
(5) *The map*
$$L^2_\nu(\Omega) \ni f \mapsto F = \mathcal{L}_\nu(f) \in \mathbf{H}_\nu(T(\Omega))$$
is a unitary isomorphism and
(6) *If* $\nu > (r-1)\frac{a}{2}$ *then the functions*
$$q_{m,\nu}(z) := \Delta(z+e)^{-\nu}\psi_m\left(\frac{z-e}{z+e}\right), \qquad m \in \Lambda,$$
form an orthogonal basis of $\mathbf{H}_\nu(T(\Omega))^L$*, the space of* L*-invariant functions in* $\mathbf{H}_\nu(T(\Omega))$*.*

The proof of (5) follows from a restriction principle argument. Let $f \in L^2_\nu(\Omega)$ be in the domain of RR^*, then

$$\begin{aligned}
RR^*f(y) &= R^*f(y)\\
&= (R^*f \mid K_y)_{\mathbf{H}_\nu(T(\Omega))}\\
&= (f, RK_y)_{L^2_\nu(\Omega)}\\
&= \int_\Omega f(x)\overline{K(x,y)}\,d\mu_\nu(x)\\
&= \Gamma_\Omega(\nu)\int_\Omega f(x)\Delta(x+y)^{-\nu}\,d\mu_\nu(x) \qquad (7.3)\\
&= \int_\Omega f(x)\mathcal{L}^\Omega(e^{-(y,\cdot)})(x)\,d\mu_\nu(x)\\
&= \int_\Omega e^{-(y,x)}\Delta(x)^{\nu-d/r}\mathcal{L}^\Omega(f)(x)\,dx\\
&= \mathcal{L}^\Omega(\mathcal{L}^\Omega f)(y),
\end{aligned}$$

and
$$(\mathcal{L}^\Omega f, \mathcal{L}^\Omega f) = (R^*f, R^*f) < \infty.$$
Thus f is in the domain of $(\mathcal{L}^\Omega)^2$ and $RR^* = (\mathcal{L}^\Omega)^2$. Therefore $(\mathcal{L}^\Omega)^2$ is a self-adjoint extension of RR^*, which is also self-adjoint by the von Neumann theorem.

Consider the inverse operator R^{-1} acting on the image of R. For a function g in the image of R, $R^{-1}g$ is the unique extension of g to a holomorphic function on $T(\Omega)$.

Thus $R^{-1}\mathcal{L}^\Omega = \mathcal{L}_\nu$. Letting R^{-1} act on the previous equality (7.3) we get

$$R^* f = \mathcal{L}_\nu \mathcal{L}^\Omega f.$$

This proves the polar decomposition formula. Since R^* is densely defined and R is an injective closed operator we have that the unitary part $\mathcal{L}_\nu$ extends to a unitary operator. Thus we get the well known fact, first proved by Rossi and Vergne [44] for the full Wallach set:

Theorem 7.2. *Let $\nu > a(r-1)/2$. Then the Laplace transform $\mathcal{L} : L^2(\Omega, d\mu_\nu) \to \mathbf{H}_\nu(T(\Omega)$*

$$\mathcal{L}(f)(z) = \int_\Omega e^{-(z,x)} f(x) \Delta(x)^{\nu - d/r}\, dx$$

is an unitary isomorphism.

Combinging Theorem 7.2 and Theorem 7.1, part 2, we get the following simple, but usefull fact:

Lemma 7.3. *The map $\Xi = \Xi_\nu : L^2(\Omega, d\mu_\nu) \to \mathbf{H}(\mathcal{D})_\nu$,*

$$\begin{aligned} \Xi(f)(w) &= \frac{1}{\sqrt{\Gamma_\Omega(\nu)}} \pi_\nu(c)^{-1} \mathcal{L}(f)(w) \\ &= \sqrt{\frac{2^{r\nu}}{\Gamma_\Omega(\nu)}} \int_\Omega e^{-(cw,x)} f(x) \Delta(x)^{\nu - d/r} dx \end{aligned}$$

is a unitary isomorphism.

8. The Laguerre Polynomials and Functions

In this section ν will stand for a complex number identified with the element $\nu(\beta_1 + \ldots + \beta_r) \in \mathfrak{a}^*$. We define, as in [15] (c.f. equation (0.3) in the Introduction), the Laguerre polynomials by

$$L_m^\nu(x) = (\nu)_m \sum_{|n| \leq |m|} \binom{m}{n} \frac{1}{(\nu)_n} \psi_n(-x)$$

and the Laguerre functions

$$\ell_m^\nu(x) = e^{-\mathrm{Tr}(x)} L_m^\nu(2x)$$

where

$$(\nu)_m = \frac{\Gamma_\Omega(\lambda + m)}{\Gamma_\Omega(\lambda)}\,.$$

Then

Theorem 8.1 ([8] Theorem 7.8)**.**

$$\mathcal{L}_\nu(\ell_m^\nu) = \Gamma_\Omega(m + \nu) q_{m,\nu}.$$

In particular, it follows that $\{\ell_m^\nu\}_m$ is an orthogonal basis for $L^2(\Omega, d\mu_\nu)^{H \cap K}$.

It follows from the fact $\|\psi_m\|^2 = \frac{\left(\frac{d}{r}\right)_m}{d_m (\nu)_m}$ and Lemma 7.3, that

Lemma 8.2.

$$\|\ell_m^\nu\|^2 = \frac{\Gamma_\Omega(\nu)\left(\frac{d}{r}\right)_m}{2^{r\nu} d_m(\nu)_m}.$$

Furthermore the action $\pi_\nu(2Z_0)$ and $\pi_\nu(\zeta)$ translates into:

Theorem 8.3 ([8], Theorem 7.9)**.**

(1) $-(\nu r+2E)\ell_m^\nu = \sum_{j=1}^r \begin{pmatrix} m \\ m-e_j \end{pmatrix} (m_j-1+\nu-\frac{a}{2}(j-1))\ell_{m-e_j}^\nu + \sum_{j=1}^r c_m(j)\ell_{m+e_j}^\nu$

(2) $\pi_\nu(\zeta)\ell_m^\nu = (r\nu + 2|m|)\ell_m^\nu$.

There are some things that remain to be done. First, one can show that $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{sl}(2,\mathbb{C}) \simeq \mathbb{C}X^- \oplus \mathbb{C}X^+ \oplus \mathbb{C}Z_0$, where $X^- \in \mathfrak{p}^+$ corresponds to an annihilating operator, $X^+ \in \mathfrak{p}^-$ corresponds to a creation operator, and $Z_0 \in \mathfrak{k}_{\mathbb{C}}$ corresponds to the Laguerre operator, when viewed as operators on $L^2(\Omega, d\mu_\nu)$. The problem is to find explicit formula as second order differential operators and the full representation of $\mathfrak{g}_{\mathbb{C}}$ on $L^2(\Omega, d\mu_\nu)$. This has been worked out for $G = SU(n,n)$[6].

Theorem 8.4 ([6], Theorem 6.1)**.** *For the cone of positive definite Hermitian matrices we have*

(1) $\mathrm{tr}(-s\nabla\nabla - v\nabla + s)\ell_m^\nu = (r\nu + |m|)\ell_m^\nu$

(2) $\frac{1}{2}\mathrm{tr}(s\nabla\nabla + (\nu I + 2s)\nabla + (vI + s)\ell_m^\nu = \sum_{j=1}^r \begin{pmatrix} m \\ m-e_j \end{pmatrix} (m_j - 1 + \nu - \frac{a}{2}(j - 1))\ell_{m-e_j}^\nu$

(3) $\frac{1}{2}\mathrm{tr}(-s\nabla\nabla + (-\nu I + 2s)\nabla + (vI - s))\ell_m^\nu = \sum_{j=1}^r c_m(j)\ell_{m+e_j}^\nu$.

These formulas generalize the classical relations

(1) $(tD^2 + \nu D - t)\ell_n^v = -(2n+\nu)\ell_n^\nu$

(2) $(tD^2 + (2t+\nu)D + (t+\nu))\ell_n^\nu = -2(n+\nu-1)\ell_{n-1}^\nu$

(3) $(tD^2 - (2t-\nu)D + (t-\nu))\ell_n^\nu = -2(n+1)\ell_{n+1}^\nu$.

9. The generating function for the Laguerre polynomials

Recall equation (0.1)

$$(1-w)^{-\alpha-1}\exp\left(\frac{xw}{w-1}\right) = \sum_{n=0}^\infty L_n^\alpha(x)w^n\,, \quad |w|<1,\ -1<\alpha\,. \tag{9.1}$$

from the introduction. Let us rewrite this using our notation $\nu = \alpha + 1$ and $\ell_m^\alpha(x) = L_m^\alpha(2x)e^{-x}$. By replacing in (9.1) x by $2x$ and multiplying both sides by $\exp(-x)$ we obtain

$$(1-w)^\nu \exp\left(x\frac{1+w}{1-w}\right) = \sum_{m=0}^\infty \ell_m^\nu(x)w^m \tag{9.2}$$

We will now show how to generalize (9.2) using the connection with the highest weight modules.

Suppose $\mathbb{H}$ is an arbitrary highest weight space for a unitary representation of G isomorphic to $\mathbf{H}_\nu$. Let $\mathbf{V} = \mathbb{H}^{\mathfrak{p}^+}$ be the lowest K-type. In other words

$$\mathbf{V} = \{v \in \mathbb{H} \mid X \cdot v = 0, \text{for all } X \in \mathfrak{p}^+\}\,.$$

In this case $\mathbf{V} = \mathbb{C}v_\circ$ is one dimensional and we can thus identify H and $\mathbb{C}$. Assume $v_\circ$ has norm 1. For each $T \in \mathfrak{p}^+$ we define a map $q_T : H \to \mathbb{H}$ by the formula

$$q_T(v) = \sum_{m=0}^{\infty} \frac{\overline{T^n} \cdot v}{n!}.$$

We have the following

Theorem 9.1 ([4] Theorems 5.1, 7.1 and 7.2)**.**

(1) *The series that defines q_T converges in $\mathbb{H}$ if and only if $T \in \mathcal{D}$.*

(2) *The map $\Xi : \mathbb{H} \to \mathbf{H}_\nu$ given by*

$$\Xi\xi(T) = q_T^*\xi, \quad T \in \mathcal{D}$$

is a unitary isomorphism that intertwines the G-actions.

(3) *In the case $\mathbb{H} = \mathbf{H}_\nu$ we have $\mathbf{V} = \mathbb{C}1$, where 1 is the constant function, and*

$$(q_T 1, F) = (1, F(T)) = \overline{F(T)}, \quad T \in \mathcal{D}.$$

In the case where $\mathbb{H} = L^2(\Omega, d\mu_\nu)$ the map Ξ is given in Lemma 7.3:

$$\begin{aligned} \Xi(f)(w) &= \frac{1}{\sqrt{\Gamma_\Omega(\nu)}} \pi_\nu(c)^{-1} \mathcal{L}(f)(w) \\ &= \sqrt{\frac{2^{r\nu}}{\Gamma_\Omega(\nu)}} \int_\Omega e^{-(cw,x)} f(x) \Delta(x)^{\nu - d/r} dx \end{aligned}$$

The highest weight space is $H = \mathbb{C}\ell_\circ^\nu$. Let $v_\circ = \sqrt{\frac{2^{r\nu}}{\Gamma_\Omega(\nu)}} \ell_\circ^\nu$. Then $v_\circ$ has norm 1 and $\Xi v_\circ$ is the constant function 1 on $\mathcal{D}$.

Theorem 9.2 ([5])**.** *Let $\mathbf{E}$ be a closed subspace of $L^2(\Omega, d\mu_\nu)$, $\{e_\alpha\}$ an orthonormal basis of $\mathbf{E}$, and $E_\alpha = \Xi(e_\alpha) \in \mathbf{H}_\nu$ for each α. Let $\mathrm{pr}_\mathbf{E}$ be the orthogonal projection of $L^2(\Omega, d\mu_\nu)$ onto $\mathbf{E}$. Then we have for $w \in \mathcal{D}$ and $x \in \Omega$:*

$$\beta_\nu \overline{\Delta(e - w)^{-\nu}} \mathrm{pr}_\mathbf{E}(e^{-\overline{(\mathbf{c}(w),x)}}) = \sum_\alpha \overline{E_\alpha(w)} e_\alpha(x),$$

where $\beta_\nu = \sqrt{\frac{2^{r\nu}}{\Gamma_\Omega(\nu)}}$ and the convergence is with respect to the Hilbert space norm in $L^2(\Omega, d\mu_\nu)$.

Proof. First we have

$$\begin{aligned} \mathrm{pr}_\mathbf{E}(q_w(v_\circ)) &= \sum_\alpha (q_w(v_\circ), e_\alpha) e_\alpha \\ &= \sum_\alpha (\Xi q_w v_\circ, \Xi e_\alpha) e_\alpha \\ &= \sum_\alpha (q_w \Xi v_\circ, E_\alpha) e_\alpha \\ &= \sum_\alpha (1, E_\alpha(w)) e_\alpha \\ &= \sum_\alpha \overline{E_\alpha(w)} e_\alpha \end{aligned}$$

On the other hand,

$$\begin{aligned}(q_w v_\circ, f) &= (\Xi q_w v_\circ, \Xi f)\\ &= (q_w 1, \Xi f)\\ &= \overline{\Xi f(w)}\\ &= \beta_\nu \overline{\Delta(e-w)^{-\nu}\int_\Omega e^{-(cw,x)} f(x)\, d\mu_\nu(x)}\\ &= \beta_\nu \overline{\Delta(e-w)^{-\nu}(e^{-(cw,x)}, f)}\end{aligned}$$

From this it follows that

$$q_w v_\circ(x) = \beta_\nu\, \overline{\Delta(e-w)^{-\nu}} e^{-\overline{(\mathbf{c}(w),x)}}.$$

and

$$\beta_\nu\, \overline{\Delta(e-w)^{-\nu}} \mathrm{pr}_{\mathbf{E}}(e^{-\overline{(\mathbf{c}(w),x)}}) = \sum_\alpha \overline{E_\alpha(w)} e_\alpha(x)\,.$$

□

We now specialize to the case where $\mathbf{E} = L^2(\Omega, d\mu_\nu)^{K\cap H}$ with orthonormal basis

$$e_m = \sqrt{\frac{2^{r\nu} d_m}{\Gamma_\Omega(\nu)\left(\frac{d}{r}\right)_m (\nu)_m}}\, \ell_m^\nu.$$

The orthogonal projection $\mathrm{pr}_{\mathbf{E}} : L^2(\Omega, d\mu_\nu) \to \mathbf{E}$ is given by

$$\mathrm{pr}_{\mathbf{E}}(f) = \int_{K\cap H} f(kx)\, dk.$$

The above theorem immediately gives:

Theorem 9.3 ([5]). *Let $w \in \mathcal{D}$ and $x \in \Omega$. Then*

$$\Delta(e-w)^{-\nu} \int_{K\cap H} e^{-(k\cdot x,(1+w)(1-w)^{-1})}\, dk = \sum_{m\in\Lambda} d_m \frac{1}{(\frac{n}{r})_m} \psi_m(w) \ell_m^\nu(x).$$

This formula is the generating function for the Laguerre functions on symmetric cones and generalizes equation (9.2) (cf [14] p.347 and the references given there).

References

[1] G. Andrews, R. Askey, and R. Roy, *Special Functions*, Encyclopedia of Mathematics and its Applications, The University Press, Cambridge, 1999.

[2] F.A. Berezin: *Connection between con- and contravariant symbols of operators on the classical complex symmetric spaces*, Dokl, Akad. Nauk USSR **241** (1978), 15–17

[3] I. Cherednik, *Nonsymmetric Macdonald polynomials*, Intern. Math. Res. Notices (1995), no. 10, 483–515.

[4] M. Davidson and R. Fabec: *Geometric realizations for highest weight representations.* In: (t. Ton-That, K.I. Gross, D.St.P. Richards, and P.J. Sally, Jr. eds) Representation Theory and Harmonic Analysis, Contemporary Mathematics **191**/ AMS 1995, pp. 13–31

[5] ______:*Laguerre functions, the generating function, and representation theory.* In preparation

[6] M. Davidson, G. Ólafsson, *Differential Recursion Relations for Laguerre Functions on Hermitian Matrices*, Integral Transforms and Special Runctions, (to appear)

[7] M. Davidson, G. Ólafsson, and Genkai Zhang, *Laguerre Polynomials, Restriction Principle, and Holomorphic Representations of* SL(2, ℝ), Acta Appl. Math. **71**(2002), no. 3, 261–277.

[8] M. Davidson, G. Ólafsson, and Genkai Zhang, ,

[9] J. Dieudonné: Special functions and linear representations of Lie groups. CBMS regional conference series in mathematics **42**, AMS, 1980

[10] G. van Dijk and S.C. Hille: *Maximal degenerate representations, Berezin kernels, and canonical representations*. In: Lie groups and Lie algebras, their representations, generalizations and applications. Kluer Academic Press, 1997

[11] G. van Dijk, and V.F. Molchanov: *Berezin forms on line bundles over complex hyperbolic spaces*, Integral Equations and Operator Theory **45** (2003), 177–230

[12] G. van Dijk and M. Pevzner, *Berezin kernels on tube domains*, J. Funct. Anal. **181** (2001), 189-208.

[13] T.J. Enright, R. Howe, and N. Wallach: *A classification of unitary highest weight modules*. In: (P.C. Trombi ed.) Representation Theory of Reductive Groups. Progress in Mathematics, **40**, Birkhäuser, Boston 1983, 97–143

[14] J. Faraut and A. Koranyi, *Function spaces and reproducing kernels on bounded symmetric domains*, J. Funct. Anal. **88** (1990), 64–89.

[15] ——, *Analysis on symmetric cones*, Oxford University Press, Oxford, 1994.

[16] S. G. Gindikin, *Analysis on homogeneous domains*, Uspekhi Mat. Nauk, **19**, 1964, 3—92; Russian Math. Surveys, **19**, no. 4, 1–89

[17] W. Groenevelt: *Laguerre functions and representations of* $\mathfrak{su}(1,1)$. Preprint, 2003, aXiv:math.CA/0302342v1

[18] W. Groenevelt, E. Koelink: *Meixner functions and polynomials related to Lie algebra representations*. J. Phys. A: Math. Gen. **35** (2002), 65–85

[19] L.I. Grosheva, and V.F. Molchanov: *Canonical and boundary representations on the Lobachevsky plane*, Acta Appl. Math. **73** (2002), 59–77

[20] Harish-Chandra: *Representations of semisimple Lie groups, VI*, Amer. J. Math. **78** (1956), 564–628

[21] S. Helgason,*Groups and Geometric Analysis*, Academic Press, 1984.

[22] J. Hilgert, and G. Ólafsson: *Analytic extensions of representations, the solvable case*, Japan. J. Math. (N.S.) **18** (1992), 213–289

[23] J. Hilgert and G. Ólafsson, *Causal Symmetric Spaces, Geometry and Harmonic Analysis*, Perspectives in Math. **18**, Academic Press, 1997

[24] H.P. Jakobsen: *Hermitian symmetric spaces and their unitary highest weight modules*, J. Funct. Anal. **52** (1983) 385–412

[25] J. Van der Jeught, R. Jagannathan: *Realizations of* $\mathfrak{su}(1,1)$ *and* $U_q(\mathfrak{su}(1,1))$ *and generating functions for orthogonal polynomials*, J. Math. Phys. **39** (1998), 5062–5078

[26] P. E. T. Jorgensen, G. Ólafsson, *Unitary Representations of Lie Groups with Reflection Symmetry*, J. Func. Anal. **158** (1998), 26-88

[27] P.E.T. Jorgensen, G. Ólafsson, *Unitary representations and Osterwalder–Schrader Duality*, In: The Mathematical Legacy of Harish-Chandra (Balt, MD, 1998), 233-401, Proc. Sympos. Pure Math., **68**, Amer. Math. Soc., Prov, RI, 2000.

[28] E. Koelink: *Spectral theory and special functions*, preprint 2001, math.CA/0107036

[29] K.T. Koelink, J. Van der Jeught: *Convolutions for orthogonal polynomials from Lie and quantum algebra representations*, SIAM J. Math. Anal. **29** (1998), 794–822

[30] A. Korányi, and J.A. Wolf: *Realizations of Hermitian symmetric spaces as generalized halfplanes*, Ann. of Math. **81** (1965), 265–288

[31] ——:*Generalized Cayley transformations of bounded symmetric domains*, Amer. J. Math. **87** (1965), 899–939

[32] O. Loos, *Bounded symmetric domains and Jordan pairs*, University of California, Irvine, 1977.

[33] D. R. Masson, J. Repka: *Spectral theory of Jacobi matrices in* $\ell^2(\mathbb{Z})$ *and the* $\mathfrak{su}(1,1)$ *Lie algebra*, SIAM J. Math. Anal. **22** (1991), 1131–1146

[34] K-H. Neeb: *Holomorphy and convexity in Lie Theory*, Expositions in Mathematics, de Gruyter, Berlin, 1999

[35] Yu. Neretin, *Matrix analogs of the integral $B(\alpha, \rho - \alpha)$ and Plancherel formula for Berezin kernel representations*, (1999), preprint, math.RT/9905045.

[36] G. Ólafsson, *Analytic Continuation in Representation Theory and Harmonic Analysis*. In: Ed. J. P. Bourguignon, T. Branson, O. Hijazi, Global Analysis and Harmonic Analysis, Seminares et Congres **4**, (2000), 201–2333. The French Math. Soc.

[37] G. Ólafsson and B. Ørsted: *The holomorphic discrete series for affine symmetric spaces, I*, J. Funct. Anal. **81** (1988), 126–159

[38] ——: *The holomorphic discrete series of an affine symmetric space and representations with reproducing kernels*, Trans. Amer. Math. Soc. **326** (1991), 385–405

[39] ——: *Generalizations of the Bargmann transform*, Lie theory and its applications in physics. Proceedings of the international workshop, Clausthal, Gemany, August 14–17, 1995. (H.-D. Doebner et al, ed.), World scientific, Singapore, 1996, pp. 3–14, MR. **99**c:22017

[40] G.I. Ol'shanskii: *Invariant cones in Lie algebras, Lie semigroups, and the holomorphic discrete series*, Functional Anal. Appl. **15** (1981), 275–285

[41] B. Ørsted and G. Zhang, *Weyl quantization and tensor products of Fock and Bergman spaces*, Indiana Univ. Math. J. **43** (1994), 551–582.

[42] ——, *Generalized principal series representations and tube domains*, Duke Math. J **78** (1995), 335–357.

[43] J. Repka, *Tensor products of holomorphic discrete series representations*, Can. J. Math. **31**(1979), 836-844.

[44] F. Ricci and A. Vignati, *Bergman spaces on some tube-type domains and Laguerre operators on symmetric cones*, J. Reine Angew. Math., **449** (1994), 81–101.

[45] H. Rossi and M. Vergne *Analytic continuation of the holomorphic discrete series of a semisimple Lie group*, Acta Math.. **136** (1976), 1-59.

[46] R.J. Stanton: *Analytic extension of the holomorphic discrete series*, Amer. J. Math. **108** (1986), 1411-1424

[47] W. Schmid, *Die Randwerte holomorpher Funktionen auf hermitesch symmetrischen Räumen*, Invent. Math **9** (1969), 61–80.

[48] A. Unterberger and H. Upmeier, *The Berezin transform and invariant differential operators*, Comm. Math. Phys. **164** (1994), 563–597.

[49] H. Upmeier, *Toeplitz operators on bounded symmetric domains*, Trans. Amer. Math. Soc. **280** (1983), 221–237.

[50] H. Upmeier, *Jordan Algebras in Analysis, Operator Theory, and Quantum Mechanics*, Regional Conference Series in Mathematics, **67**, AMS 1987

[51] N.J. Vilenkin: Special Functions and the Theory of Group Representations. Translations of Mathematical Monographs, **32**, AMS , Rhode Island, 1968

[52] N.J. Vilenkin, A.U. Klimyk: Representations of Lie groups and special functions, Vol 1. Kluwer Academi Publisher, Dorderecht, 1991

[53] N. Wallach, *The analytic continuation of the discrete series, I, II*, Trans. Amer. Math. Soc. **251** (1979), 1-17; 19-37.

[54] G. Zhang, *Tensor products of weighted Bergman spaces and invariant Ha-plitz operators,* Math. Scand., **71**(1992), 85-95.

[55] G. Zhang, *Tensor products of minimal holomorphic representations* , Representation Theory, **5**(2001), 164-190.

[56] G. Zhang, *Berezin transform on real bounded symmetric domains*, Trans. Amer. Math. Soc., **353** (2001), no. 9, 3769–3787

[57] G. Zhang, *Branching coefficients of holomorphic representations and Segal-Bargmann transform* , J. Functional Analysis, **195** (2002), no. 2, 306-349.

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA

E-mail address: davidson@math.lsu.edu

E-mail address: olafsson@math.lsu.edu